A Liouville-type theorem for Schrödinger operators

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Abstract

In this paper we prove a sufficient condition, in terms of the behavior of a ground state of a symmetric critical operator P_1, such that a nonzero subsolution of a symmetric nonnegative operator P_0 is a ground state. Particularly, if $P_j := -\Delta + V_j$, for $j = 0, 1$, are two nonnegative Schrödinger operators defined on $\Omega \subseteq \mathbb{R}^d$ such that P_1 is critical in Ω with a ground state φ, the function $\psi \not\leq 0$ is a subsolution of the equation $P_0 u = 0$ in Ω and satisfies $|\psi| \leq C\varphi$ in Ω, then P_0 is critical in Ω and ψ is its ground state. In particular, ψ is (up to a multiplicative constant) the unique positive supersolution of the equation $P_0 u = 0$ in Ω. Similar results hold for general symmetric operators, and also on Riemannian manifolds.

2000 *Mathematics Subject Classification.* Primary 35J10; Secondary 35B05.

Keywords. Green function, ground state, Liouville theorem, positive solution.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a domain. We assume that $A : \Omega \to \mathbb{R}^{d^2}$ is a measurable matrix valued function such that for every compact set $K \subset \Omega$ there exists

$\mu_K > 1$ such that

$$\mu_K^{-1} I_d \le A(x) \le \mu_K I_d \qquad \forall x \in K, \tag{1.1}$$

where I_d is the d-dimensional identity matrix, and the matrix inequality $A \le B$ means that $B - A$ is a nonnegative matrix on $\mathbb{R}^d$. Let $V \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R})$, where $p > d/2$. We consider the quadratic form

$$\mathbf{a}_{A,V}[u] := \int_\Omega \left(A\nabla u \cdot \nabla u + V|u|^2 \right) \mathrm{d}x \tag{1.2}$$

on $C_0^\infty(\Omega)$ associated with the Schrödinger equation

$$Pu := (-\nabla \cdot (A\nabla) + V)u = 0 \qquad \text{in } \Omega. \tag{1.3}$$

We say that $\mathbf{a}_{A,V}$ is *nonnegative* on $C_0^\infty(\Omega)$, if $\mathbf{a}_{A,V}[u] \ge 0$ for all $u \in C_0^\infty(\Omega)$.

Definition 1.1. We say that $v \in H^1_{\mathrm{loc}}(\Omega)$ is a *(weak) solution* of (1.3) if for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (A\nabla v \cdot \nabla \varphi + Vv\varphi)\, \mathrm{d}x = 0. \tag{1.4}$$

We say that $v \in H^1_{\mathrm{loc}}(\Omega)$ is a *subsolution* of (1.3) if for every nonnegative $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (A\nabla v \cdot \nabla \varphi + Vv\varphi)\, \mathrm{d}x \le 0. \tag{1.5}$$

$v \in H^1_{\mathrm{loc}}(\Omega)$ is a *supersolution* of (1.3) if $-v$ is a subsolution of (1.3).

Let $\mathcal{C}_P(\Omega)$ be the cone of all positive solutions of the equation $Pu = 0$ in Ω, and let

$$\lambda_0(P, \Omega) := \sup\{\lambda \in \mathbb{R} \mid \mathcal{C}_{P-\lambda}(\Omega) \neq \emptyset\} \tag{1.6}$$

be the *generalized principal eigenvalue* of the operator P in Ω. By the Allegretto-Piepenbrink theory (see for example, [1, 21]), the form $\mathbf{a}_{A,V}$ is nonnegative on $C_0^\infty(\Omega)$ if and only if $\lambda_0(P, \Omega) \ge 0$.

Let $K \Subset \Omega$ (i.e. K is relatively compact in Ω). Recall [1, 21] that $u \in \mathcal{C}_P(\Omega \setminus K)$ is said to be a *positive solution of the operator P of minimal growth in a neighborhood of infinity in* Ω, if for any $K \Subset K_1 \Subset \Omega$, with a smooth boundary, and any $v \in \mathcal{C}_P(\Omega \setminus K_1) \cap C((\Omega \setminus K_1) \cup \partial K_1)$, the inequality $u \le v$ on ∂K_1 implies that $u \le v$ in $\Omega \setminus K_1$. A positive solution $u \in \mathcal{C}_P(\Omega)$

which has minimal growth in a neighborhood of infinity in Ω is called a *ground state* of P in Ω.

The operator P is said to be *critical* in Ω, if P admits a ground state in Ω. The operator P is called *subcritical* in Ω, if $\mathcal{C}_P(\Omega) \neq \emptyset$, but P is not critical in Ω. If $\mathcal{C}_P(\Omega) = \emptyset$, then P is *supercritical* in Ω.

It is known that the operator P is critical in Ω if and only if the equation $Pu = 0$ in Ω admits (up to a multiplicative constant) a unique positive supersolution. In particular, in the critical case we have $\dim \mathcal{C}_P(\Omega) = 1$ (see for example [18, 21] and the references therein).

On the other hand, P is subcritical in Ω, if and only if P admits a *positive minimal Green function* $G_P^\Omega(x, y)$ in Ω. For each $y \in \Omega$, the function $G_P^\Omega(\cdot, y)$ is a positive solution of the equation $Pu = 0$ in $\Omega \setminus \{y\}$ that has minimal growth in a neighborhood of infinity in Ω and has a (suitably normalized) nonremovable singularity at y (see for example [18, 21] and the references therein).

The following basic example will be used few times along the paper.

Example 1.2. Let $P = -\Delta$ and $\Omega = \mathbb{R}^d$. It is well known that $\lambda_0(-\Delta, \mathbb{R}^d) = 0$. In addition, the positive Liouville theorem asserts that

$$\mathcal{C}_{-\Delta}(\mathbb{R}^d) = \{c\mathbf{1} \mid c > 0\},$$

where $\mathbf{1}$ is the constant function taking at any point the value 1. Moreover, $-\Delta$ is critical in $\mathbb{R}^d$ if and only if $d \leq 2$.

Recently, Berestycki, Hamel, and Roques [6] has introduced the following definition which arises naturally in the study of some semilinear equations.

Definition 1.3.

$$\lambda_0'(P, \Omega) := \inf\{\lambda \in \mathbb{R} \mid \exists \phi \in C^2(\Omega) \cap W^{2,\infty}(\Omega), \phi > 0, (P - \lambda)\phi \leq 0 \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega, \text{ if } \partial\Omega \neq \emptyset\}.$$

Before formulating our results, we present four basic problems (Problems 1–4) concerning the particular case $\Omega = \mathbb{R}^d$, which have been solved in the past few years. It turns out that (almost) all these results follow directly from our main result (Theorem 1.7).

Problem 1 ([23, 24]). *Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Does the existence of a positive bounded solution to the equation*

$$H_V u := (-\Delta + V)u = 0 \qquad \text{on } \mathbb{R}^d \tag{1.7}$$

imply that H_V is critical in $\mathbb{R}^d$?

Problem 2 ([5]). *Suppose that V is smooth and bounded. Does the existence of a sign-changing bounded solution to equation* (1.7) *imply that $\lambda_0(H_V, \mathbb{R}^d) < 0$?*

Problem 3 ([5, 14]). *Let σ be a strictly positive C^2-function on $\mathbb{R}^d$, and consider the divergence form operator $L = \nabla \cdot (\sigma^2 \nabla)$ on $\mathbb{R}^d$. Suppose that the equation $L\psi = 0$ in $\mathbb{R}^d$ admits a nonzero solution ψ such that $\psi\sigma$ is bounded. Is ψ necessarily the constant function?*

Problem 4 ([7, Conjecture 4.6]). *Suppose that $P = -\nabla \cdot (A\nabla) + V$ is uniformly elliptic operator with smooth bounded coefficients on $\mathbb{R}^d$. Does the inequality*

$$\lambda_0(P, \mathbb{R}^d) \leq \lambda_0'(P, \mathbb{R}^d)$$

holds true in any dimension d.

The answers to the above four problems for the free Laplacian in $\mathbb{R}^d$ are well known. Nevertheless, the above problems are not of perturbational nature since there is no assumption on the asymptotic behavior of the coefficients of the given operator near infinity.

Problem 1 was posed by B. Simon in [23, 24]. Clearly, the answer to Problem 1 is 'no' for $d \geq 3$. Partial results concerning Problem 1 for $d \leq 2$ were obtained under the assumption that V is a short-range potential (see for example, [12, 13, 16, 18]). On the other hand, Gesztesy, and Zhao showed in [12, Example 4.6] that there is a critical Schrödinger operator on $\mathbb{R}$ with 'almost' short-range potential such that its ground state behaves logarithmically.

In a recent article Damanik, Killip, and Simon proved a result which reveals a complete answer to Problem 1.

Theorem 1.4 (cf. [10, Theorem 5]). *The answer to Problem 1 is "yes" if and only if $d = 1, 2$.*

Indeed, for $d = 1, 2$, it is shown in [10] that if the equation $H_V u = 0$ admits a positive bounded solution, then any $W \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying $H_{V \pm W} \geq 0$ is necessarily the zero potential. But this property holds if and only if H_V is critical (see [18]).

Let us turn to Problem 2 which was raised by Berestycki, Caffarelli, and Nirenberg [5]. This problem is closely related to De Giorgi's conjecture [11] (see [3, 4, 5, 14]). In [14], Ghoussoub and Gui showed a connection between Problem 2 and Problem 3 which concerns the Liouville property for operators in divergence form (see also the proof of Theorem 1.7 in [5]). In fact, the following result is proved in [5, 14, 3].

Theorem 1.5. *The answers to problems 2 and 3 are "yes" if and only if* $d = 1, 2$.

Note that Ghoussoub and Gui [14] used this Liouville-type theorem for $d = 2$ [5], to prove De Giorgi's Conjecture in $\mathbb{R}^2$.

Problem 3 was posed by Berestycki and Rossi [7] who also solved it for $d \leq 3$:

Theorem 1.6 ([7, Theorem 4.1]). *Suppose that* $P = -\nabla \cdot (A\nabla) + V$ *is uniformly elliptic operator with smooth bounded coefficients on* $\mathbb{R}^d$*. If* $d \leq 3$*, then*

$$\lambda_0(P, \mathbb{R}^d) \leq \lambda_0'(P, \mathbb{R}^d).$$

The purpose of the present article is to (partially) generalize theorems 1.4, 1.5, and 1.6 to general symmetric operators which are defined on an arbitrary domain $\Omega \subseteq \mathbb{R}^d$, or on a noncompact Riemannian manifold. Our main result is as follows.

Theorem 1.7. *Let* Ω *be a domain in* $\mathbb{R}^d$, $d \geq 1$*. Consider two Schrödinger operators defined on* Ω *of the form*

$$P_j := -\nabla \cdot (A_j \nabla) + V_j \qquad j = 0, 1, \tag{1.8}$$

such that $V_j \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R})$ *for some* $p > d/2$*, and* A_j *satisfy* (1.1).

Assume that the following assumptions hold true.

(i) The operator P_1 *is critical in* Ω*. Denote by* $\varphi \in \mathcal{C}_{P_1}(\Omega)$ *its ground state.*

(ii) $\lambda_0(P_0, \Omega) \geq 0$*, and there exists a real function* $\psi \in H^1_{\mathrm{loc}}(\Omega)$ *such that* $\psi_+ \neq 0$*, and* $P_0\psi \leq 0$ *in* Ω*, where* $u_+(x) := \max\{0, u(x)\}$*.*

(iii) The following matrix inequality holds

$$(\psi_+)^2(x)A_0(x) \leq C\varphi^2(x)A_1(x) \qquad a.\ e.\ in\ \Omega, \tag{1.9}$$

where $C > 0$ *is a positive constant.*

Then the operator P_0 *is critical in* Ω*, and* ψ *is its ground state. In particular,* $\dim \mathcal{C}_{P_0}(\Omega) = 1$ *and* $\lambda_0(P_0, \Omega) = 0$*.*

Corollary 1.8. *Suppose that all the assumptions of Theorem 1.7 are satisfied except possibly the assumption that* $\lambda_0(P_0, \Omega) \geq 0$*. Assume further that either* ψ *changes its sign in* Ω*, or* ψ *is not a solution of the equation* $P_0 u = 0$ *in* Ω*. Then* $\lambda_0(P_0, \Omega) < 0$*.*

Theorem 1.7 and Corollary 1.8 imply in particular the sufficiency parts of Theorem 1.4 and Theorem 1.5, and also Theorem 1.6 for $d = 1, 2$. Note that in contrast to the assumptions of theorem 1.5 and 1.6, we assume in Theorem 1.7 neither that the functions V_j are bounded and smooth, nor that the matrix valued functions A_j are smooth.

The outline of the paper is as follows. In Section 2, we present some results from [19] that will be used in the proof of Theorem 1.7. Section 3 is devoted to the proof of Theorem 1.7 and its consequences. We conclude the paper in Section 4, where we pose two open problems suggested by the results of the present paper.

2 Preliminary results

Definition 2.1. We say that a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ is a *null sequence* for the nonnegative quadratic form $\mathbf{a}_{A,V}$ if there exists an open set $B \Subset \Omega$ such that $\int_B |u_k|^2 \,\mathrm{d}x = 1$, and

$$\lim_{k\to\infty} \mathbf{a}_{A,V}[u_k] = 0. \tag{2.1}$$

We say that a positive function φ is a *null state* for the nonnegative quadratic form $\mathbf{a}_{A,V}$, if there exists a null sequence $\{u_k\}$ for the form $\mathbf{a}_{A,V}$ such that $u_k \to \varphi$ in $L^2_{\mathrm{loc}}(\Omega)$.

Remark 2.2. The requirement that $u_k \subset C_0^\infty(\Omega)$, can clearly be weakened by assuming only that $\{u_k\} \subset H_0^1(\Omega)$. Also, the requirement that $\int_B |u_k|^2 \,\mathrm{d}x = 1$ can be replaced by $\int_B |u_k|^2 \,\mathrm{d}x \asymp 1$, where $f_k \asymp g_k$ means that there exists a positive constant C such that $C^{-1} g_k \leq f_k \leq C g_k$ for all $k \in \mathbb{N}$.

The following auxiliary lemma is well known (see, e.g. [9, 17, 19]).

Lemma 2.3. *Let $\psi \in H^1_{\mathrm{loc}}(\Omega)$ be a nonnegative subsolution of the equation $P\psi = 0$ in Ω. Then for any nonnegative $v \in C_0^\infty(\Omega)$ we have*

$$\mathbf{a}_{A,V}[\psi v] \leq \int_\Omega (\psi)^2 A\nabla v \cdot \nabla v \,\mathrm{d}x. \tag{2.2}$$

Moreover, if ψ is a (real valued) solution of the equation $P\psi = 0$ in Ω, then for any $v \in C_0^\infty(\Omega)$ we have

$$\mathbf{a}_{A,V}[\psi v] = \int_\Omega (\psi)^2 A\nabla v \cdot \nabla v \,\mathrm{d}x. \tag{2.3}$$

Proof. Follows from the definition of a weak (sub)solution and elementary calculation. □

The following theorem was recently proved by K. Tintarev and the author [19] (see also [20]).

Theorem 2.4. *Suppose that $\mathbf{a}_{A,V} \geq 0$ on $C_0^\infty(\Omega)$. Then $\mathbf{a}_{A,V}$ has a null sequence if and only if the corresponding operator $P = -\nabla \cdot (A\nabla) + V$ is critical in Ω. In this case, any null sequence converges in $L^2_{\mathrm{loc}}(\Omega)$ to $c\varphi$, where φ is a ground state of the operator P and c is a nonzero constant.*

Moreover, there exists a null sequence $\{u_k\}$ of nonnegative functions that converges to φ locally uniformly in $\Omega \setminus \{x_0\}$, where x_0 is some point in Ω.

3 Proof of Theorem 1.7

In this section we prove Theorem 1.7 and some consequences.

Proof of Theorem 1.7. Since ψ satisfy $P_0\psi \leq 0$ in Ω, it follows that $P_0\psi_+ \leq 0$ in Ω (see for example [1, Lemma 2.7]).

By Theorem 2.4 and our assumptions, there exists a null sequence $\{u_k\}$ for the quadratic form $\mathbf{a}_{A_1,V_1}$ of nonnegative functions which converges locally

uniformly in $\Omega\setminus\{x_0\}$ and in $L^2_{\mathrm{loc}}(\Omega)$ to the ground state φ of the operator P_1, and satisfies $\int_B (u_k)^2\,\mathrm{d}x = 1$ for some open set $B \Subset \Omega\setminus\{x_0\}$ and all $k\in\mathbb{N}$.

Denote $w_k := u_k/\varphi$. Since $w_k \to$ constant locally uniformly in $\Omega\setminus\{x_0\}$ and $\psi_+ \neq 0$, it follows that $\int_{B_1}(\psi_+ w_k)^2\,\mathrm{d}x \asymp 1$ for some open set $B_1 \Subset \Omega$ and every $k \geq k_0$. Moreover, by Lemma 2.3 and our assumptions, we have

$$\mathbf{a}_{A_0,V_0}[\psi_+ w_k] \leq \int_\Omega (\psi_+)^2 A_0 \nabla w_k\cdot\nabla w_k\,\mathrm{d}x \leq$$

$$C\int_\Omega \varphi^2 A_1\nabla w_k\cdot\nabla w_k\,\mathrm{d}x = C\mathbf{a}_{A_1,V_1}[\varphi w_k] = C\mathbf{a}_{A_1,V_1}[u_k] \to 0. \quad (3.1)$$

Therefore, $\{\psi_+ w_k\}$ is a null sequence for P_0. By Theorem 2.4, P_0 is critical in Ω and ψ_+ is its ground state. In particular, ψ_+ is strictly positive, and hence $\psi_- = 0$, and $\psi = \psi_+$ is the ground state of P_0. $\square$

Remark 3.1. Suppose that all the assumptions of Theorem 1.7 are satisfied except possibly the assumption that $\lambda_0(P_0,\Omega)\geq 0$. One can show directly that $\lambda_0(P_0,\Omega)\leq 0$. Indeed, using the notations of the proof of Theorem 1.7, we have that for some $C_1 > 0$

$$\int_\Omega (\psi_+ w_k)^2\,\mathrm{d}x \geq C_1\int_B (u_k)^2\,\mathrm{d}x = C_1 \qquad \forall k\geq k_0.$$

Moreover, by Lemma 2.3 and our assumptions, we have

$$\frac{\mathbf{a}_{A_0,V_0}[\psi_+ u_k]}{\int_\Omega(\psi_+ w_k)^2\,\mathrm{d}x} \leq \frac{\int_\Omega(\psi_+)^2 A_0\nabla w_k\cdot\nabla w_k\,\mathrm{d}x}{\int_\Omega(\psi_+ w_k)^2\,\mathrm{d}x} \leq$$

$$\tilde{C}\frac{\int_\Omega \varphi^2 A_1\nabla w_k\cdot\nabla w_k\,\mathrm{d}x}{\int_B (u_k)^2\,\mathrm{d}x} = \tilde{C}\mathbf{a}_{A_1,V_1}[\varphi w_k] = \tilde{C}\mathbf{a}_{A_1,V_1}[u_k]\to 0. \quad (3.2)$$

Therefore, the Rayleigh-Ritz variational formula implies that $\lambda_0(P_0,\Omega)\leq 0$.

It follows that

$$\begin{aligned}\lambda_0(P_0,\Omega) \leq \inf\{\lambda\in\mathbb{R} \quad | \quad & \exists\,\psi \lneqq 0, (P_0-\lambda)\psi\leq 0 \text{ in } \Omega \text{ s.t.}\\ & \psi^2(x)A_0(x)\leq C\varphi^2(x)A_1(x) \text{ in } \Omega \text{ for some}\\ & \text{critical operator } P_1 \text{ with a ground state } \varphi\}.\end{aligned}$$

In particular, if $P = -\nabla\cdot(A\nabla)+V$ is an elliptic operator on $\mathbb{R}^d$, $d\leq 2$, with a bounded matrix A, then $\lambda_0(P,\mathbb{R}^d)\leq \lambda_0'(P,\mathbb{R}^d)$ (cf. Theorem 1.6).

Recall that if $P := -\nabla \cdot (A\nabla) + V$ is $\mathbb{Z}^d$-periodic on $\mathbb{R}^d$, then $P - \lambda_0$ admits a (unique) periodic positive solution (see for example [15, 21]). On the other hand, $-\Delta$ is critical in $\mathbb{R}^d$ if and only if $d \leq 2$ (see Example 1.2). Therefore, Theorem 1.7 implies the following result of R. Pinsky (who proved it for general second-order elliptic $\mathbb{Z}^d$-periodic operators).

Corollary 3.2 ([22]). *Assume that the coefficients of the elliptic operator $P := -\nabla \cdot (A\nabla) + V$ are $\mathbb{Z}^d$-periodic on $\mathbb{R}^d$. Then the operator $P - \lambda_0$ is critical in $\mathbb{R}^d$ if and only if $d \leq 2$.*

Remark 3.3. Suppose that P_j are two nonnegative symmetric operators which are defined on a noncompact Riemannian manifold M of dimension d, where $j = 0, 1$. Since Lemma 2.3 holds true also in this case (see [17]), it follows that Theorem 2.4 is valid on Riemannian manifolds, which in turn implies that Theorem 1.7 holds true also in this case.

Recall that a Riemannian manifold M is called *recurrent* if the Laplace-Beltrami operator on M is critical (see [21]). Therefore, we have in particular, the following generalization of Theorem 1.4 and Theorem 1.5.

Theorem 3.4. *Let M be a recurrent Riemannian noncompact manifold of dimension d. Let $V \in L^2_{\mathrm{loc}}(M)$. Suppose that $H_V := -\Delta + V \geq 0$ on $C_0^\infty(M)$, and that the equation $H_V u = 0$ in M admits a nonzero bounded subsolution ψ such that $\psi_+ \neq 0$. Then H_V is critical in M and ψ is a ground state of H_V in M. In particular, $\lambda_0(H_V) = 0$, the space of all bounded solutions of the equation $H_V u = 0$ in M is one-dimensional, and $\dim \mathcal{C}_{H_V}(M) = 1$.*

In addition, one can use the results in [15] and [8, Theorem 5.2.11] to extend Corollary 3.2 to the case of equivariant Schrödinger operators on cocompact nilpotent coverings.

Corollary 3.5. *Let M be a noncompact nilpotent covering of a compact Riemannian manifold of dimension d. Suppose that $P := -\Delta + V$ is an equivariant operator on M with respect to the (nilpotent) deck group G. Then $P - \lambda_0$ is critical in M if and only if G has a normal subgroup of finite index isomorphic to $\mathbb{Z}^n$ for $n \leq 2$.*

Theorem 1.7 can be considered as a sufficient condition for the removability of singularity at infinity in Ω or as a Phragmén-Lindelöf type principle. A positive solution of (1.3) in $\Omega \setminus K$, where $K \Subset \Omega$, is called *singular at infinity* if it does not have minimal growth in a neighborhood of infinity in Ω.

Accordingly, Theorem 1.7 implies that the behavior of a positive solution of minimal growth in a neighborhood of infinity in Ω of an equation of the form (1.8), dictates some 'growth' on all positive singular at infinity solutions of *any* equation of the form (1.8). More precisely, we have the following result.

Corollary 3.6. *Suppose that for $j = 0, 1$ the operators P_j are of the form* (1.8)*, and A_j satisfy* (1.1)*. Let u_1 be a positive solution of the equation $P_1 u = 0$ of minimal growth in a neighborhood of infinity in Ω, and let u_0 be a positive solution of the equation $P_0 u = 0$ in $\Omega \setminus K$, where $K \Subset \Omega$. If $(u_0)^2 A_0 \leq C(u_1)^2 A_1$ in $\Omega \setminus K$, then u_0 is nonsingular at infinity, i.e., u_0 is a positive solution of the equation $P_0 u = 0$ of minimal growth in a neighborhood of infinity in Ω.*

Proof of Corollary 3.6. Let $\widetilde{u_0}, \widetilde{u_1} \in H^1_{\mathrm{loc}}(\Omega)$ be positive functions which are defined in Ω such that $\widetilde{u_j}|_{\Omega \setminus K_1} = u_j$, and $\widetilde{u_j}|_{\overline{K_1}}$ are sufficiently smooth, where $K_1 \Subset \Omega$, and $j = 0, 1$.

Then for $j = 0, 1$, $\widetilde{u_j} \in \mathcal{C}_{\widetilde{P_j}}(\Omega)$, where the operators $\widetilde{P_j}$ are of the form (1.8) and satisfy $\widetilde{P_j}|_{\Omega \setminus K_2} = P_j$ for some $K_2 \Subset \Omega$. Since u_1 (and hence also $\widetilde{u_1}$) is a positive solution of the equation $\widetilde{P_1} u = 0$ of minimal growth in a neighborhood of infinity in Ω, it follows that $\widetilde{u_1}$ is a ground state of the critical operator $\widetilde{P_1}$ in Ω. Therefore, Theorem 1.7 implies that $\widetilde{u_0}$ is a ground state of the critical operator $\widetilde{P_0}$ in Ω. Hence, u_0 is a positive solution of the equation $P_0 u = 0$ of minimal growth in a neighborhood of infinity in Ω. $\square$

Example 3.7. Let $d \geq 2$, and $V \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, where $p > d/2$. Suppose that $H_V := -\Delta + V \geq 0$ on $C_0^\infty(\mathbb{R}^d)$, and the equation $H_V u = 0$ on $\mathbb{R}^d$ has a subsolution solution $\psi \not\leq 0$ satisfying

$$\psi_+(x) = O(|x|^{\frac{2-d}{2}}) \qquad \text{as } |x| \to \infty. \tag{3.3}$$

Since $\varphi(x) := |x|^{\frac{2-d}{2}}$ is a positive solution of the Hardy-type equation

$$-\Delta u - \left(\frac{d-2}{2}\right)^2 \frac{u}{|x|^2} = 0$$

of minimal growth in a neighborhood of infinity in $\mathbb{R}^d$, it follows from Corollary 3.6 that H_V is critical in $\mathbb{R}^d$ and ψ is its ground state (cf. Theorem 1.7 in [5]).

Example 3.8. Let $d = 1$, and $V \in L^p_{\mathrm{loc}}(\mathbb{R})$, where $p > 1$. Suppose that $H_V := -\mathrm{d}^2/\mathrm{d}x^2 + V \geq 0$ on $C_0^\infty(\mathbb{R})$, and the equation $H_V u = 0$ on $\mathbb{R}$ has a subsolution solution $\psi \not\leq 0$ satisfying

$$\psi_+(x) = O(\log |x|) \qquad \text{as } |x| \to \infty. \tag{3.4}$$

It follows from [12, Example 4.6] and Corollary 3.6 that H_V is critical in $\mathbb{R}$ and ψ is its ground state.

4 Open problems

We conclude the paper with two open problems suggested by the above results which are left for future investigation.

Problem 5. *Generalize Theorem 1.7 to the class of nonsymmetric second-order linear elliptic operators with real coefficients which have the same (or even comparable) principal parts, or at least to the subclass of operators which differ only by their zero-order terms.*

Remarks 4.1. 1. Clearly, the condition (1.9) which involves the *squares* of the corresponding solutions of the symmetric operators P_j, for $j = 0, 1$, should be replaced in the nonsymmetric case by a condition which involves *products* of the form $u_j u_j^*$, where u_j (resp. u_j^*) are the corresponding solutions of the operators P_j (resp. of the formal adjoint operators P_J^*) for $j = 0, 1$.

2. Let u be a positive solution of an equation of the form (1.3) of minimal growth in a neighborhood of infinity in Ω, then Corollary 3.6 implies that any positive solution v of another equation of the form (1.3) (with a comparable principal part) in a neighborhood of infinity in Ω which is singular at infinity satisfies

$$\liminf_{x \to \infty} \frac{u(x)}{v(x)} = 0$$

in the one-point compactification of Ω (∞ denotes the point at infinity in Ω).

Ancona proved [2] that a subcritical symmetric (or even quasi-symmetric) operator P in Ω always admits $v \in \mathcal{C}_P(\Omega)$, such that

$$\lim_{x \to \infty} \frac{G_P^\Omega(x, x_0)}{v(x)} = 0.$$

Moreover, it is shown in [2] that such a positive solution does not always exist for general nonsymmetric operators. This result indicates that the answer to Problem 5 in the nonsymmetric case might be more involved.

The second problem that we pose deals with Liouville-type theorems for the p-Laplacian with a potential term. Let Ω be a domain in $\mathbb{R}^d$, $d \geq 2$, and $1 < p < \infty$. Fix $V \in L^\infty_{\mathrm{loc}}(\Omega)$. Recently the criticality theory for linear equations was extended in [20] to quasilinear equations of the form

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) + V|u|^{p-2}u = 0 \qquad \text{in } \Omega. \tag{4.1}$$

In particular, Theorem 2.4 was proved also for such equations. Therefore, it is natural to pose the following problem.

Problem 6. *Assume that* $1 < p \leq d$. *Generalize Theorem 1.7 to positive solutions of quasilinear equations of the form* (4.1).

Acknowledgments

The author wishes to thank H. Brezis, F. Gesztesy, and M. Marcus for valuable discussions. This work was partially supported by the RTN network "Nonlinear Partial Differential Equations Describing Front Propagation and Other Singular Phenomena", HPRN-CT-2002-00274, the Israel Science Foundation founded by the Israeli Academy of Sciences and Humanities, and the Fund for the Promotion of Research at the Technion.

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